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'LETTERS' SECTION

On the Statistics Consistent with Nambu's New Quantum Rules. Quarks?

MÁXIMO GARCÍA SUCRE and ANDRÉS J. KÁLNAY

Centro de Física, Instituto Venezolano de Investigaciones Científicas (IVIC), Apartado 1827, *Caracas* 101, *Venezuela*

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Nambu (1973) proposed a new classical mechanics.‡ For one of its quantum forms he introduced what we call the Nambu Quantum Bracket $[A, B, C] = [A, B]_- C + [B, C]_- A + [C, A]_- B$, where $[A, B]_- = AB - BA$, and stated the quantum rule

$$[X_1, X_2, X_3] = iI \tag{1}$$

for a three-component phase space variable $\mathbf{X} = (X_1, X_2, X_3)$. Our aim is to show that in a second quantised theory and in a sense to be cleared below, the main statistics used in quantum physics are consistent with the quantum rule (1). This is not *a priori* obvious because a quantum rule generally severely restricts the possible statistics. For example, the quantum rule $[q, p]_- = iI$ conducts to bosons in a second quantised theory via $b = 2^{-1/2}(q + ip)$, but is inconsistent with Fermi statistics which require anticommutation rules. We shall compare a quantum Nambu system of coordinates $\mathbf{X}$ with a second quantised system with only one annihilation operator called b for the Bose or para-Bose (= pB) case,§ f_ϵ with $\epsilon = +1$ for the Fermi or para-Fermi (= pF)§ case and with $\epsilon = -1$ for the modified para-Fermi (mpF) case.|| We work with single vacuum irreducible representations of parastatistics, calling p the order.

‡ See also Bell & Nambu (to be published). Cohen & Kálnay (1975).

§ Green (1953), Greenberg & Messiah (1965).

|| Kademova & Kraev (1971a, b), Kamefuchi & Takahashi (1962). Indefinite metric or non-Fock representations must be used when mpF has several pairs of generators (Bracken & Gray, 1971; Ohnuki, Yamada & Kamefuchi, 1971; Kademova & Kraev, 1972; Geyer, 1973).

Proposition 1. Let f_ϵ be a pF or mpF annihilation operator. Let us define

$$X_1 = K(f_\epsilon + f_\epsilon^\dagger), \qquad X_2 = iK(f_\epsilon - f_\epsilon^\dagger), \qquad X_3 = \epsilon K[f_\epsilon^\dagger, f_\epsilon]_- \tag{2a}$$

$$K = \tfrac{1}{2}\{\tfrac{1}{2}[f_\epsilon, f_\epsilon^\dagger]_+ + \epsilon\,\tfrac{1}{4}[f_\epsilon, f_\epsilon^\dagger]_-^2\}^{-1/3} \tag{2b}$$

and exclude $p = 2$ for mpF. Then **X** generates a realisation of Nambu algebra defined by equation (1) in terms of the pF ($\epsilon = +1$) (respectively mpF, i.e. $\epsilon = -1$) algebra.

Proof. Nambu has shown a $SO(3)$ (and also a $SO(2,1)$) realisation of his algebra and it is known¶ that the algebra of the pF (respectively mpF) quantisation with only one annihilation operator f_ϵ is the $SO(3)$ (respectively $SO(2,1)$) algebra. This implies the existence of the realisation. Equations (2) follow from the comparison of the formulae of Nambu for the case $SO(3)$ (respectively $SO(2, 1)$) with those of Ryan & Sudarshan (1963) (respectively Kademova & Kraev (1971b)). Notice $\{\ \} = \frac{1}{2}p(1 + \frac{1}{2}\epsilon p) \neq 0$. □

Corollary 2. Equations (2) offer a Fermi realisation of Nambu algebra.

Proof. The Fermi algebra is an irreducible representation ($p = 1$) of the pF one.§

Proposition 3. Let **X** be the phase space vector of a $SO(3)$ (or $SO(2,1)$) realisation of the Nambu algebra defined by equation (1). Let us define

$$f_\epsilon = (X_1^2 + X_2^2 + \epsilon X_3^2)(X_1 - iX_2), \qquad f_\epsilon^\dagger = (X_1 + iX_2)(X_1^2 + X_2^2 + \epsilon X_3^2) \tag{3}$$

Then $f_\epsilon, f_\epsilon^\dagger$ are the annihilation and creation operators of the pF (respectively mpF) algebra.

Proof. Same as for Proposition 1. □

Remark 4. The pF vacuum conditions are equivalent in Nambu algebra to

$$(X_1 - iX_2)\,|0\rangle = 0 \qquad \text{and} \qquad X_3\,|0\rangle = -(p/2)^{2/3}[1 + (p/2)]^{-1/3}\,|0\rangle \tag{4}$$

Proposition 5. Let b be a pB annihilation operator. Let

$$X_1 = ik(b^{\dagger 2} - b^2), \qquad X_2 = k(b^{\dagger 2} + b^2), \qquad X_3 = k[b, b^\dagger]_+ \tag{5.a}$$

$$k = \tfrac{1}{2}\{[b^2, b^{\dagger 2}]_+ - \tfrac{1}{2}[b, b^\dagger]_+^2\}^{-1/3} \tag{5.b}$$

and exclude $p = 4$. Then **X** generates a realisation of Nambu algebra defined by equation (1) in terms of the pB algebra.

Proof. From Jordan, Mukunda & Pepper (1963) we learn that the generators of $SO(2,1)$ algebra can be constructed in terms of those of the pB algebra with only two generators $b, b^\dagger$. Nambu has shown an $SO(2,1)$ realisation of his algebra. Notice $\{\ \} = p(2 - \frac{1}{2}p) \neq 0$. □

§ Green (1953), Greenberg & Messiah (1965).

¶ Ryan & Sudarshan (1963); Jordan, Mukunda & Pepper (1963); Kamefuchi & Takahashi (1962); Kademova & Kraev (1971b).

Corollary 6. Equation (5) offers a Bose realisation of Nambu algebra.

Proof. The Bose algebra is an irreducible representation ($p = 1$) of the pB one.§

Remark 7. We are not showing for pB a proposition like number 3. But such a proposition holds for some specific representations of Nambu algebra.

Remark 8. Let us assume that Propositions 1 and 3 can be extended to several degrees of freedom. Greenberg (1964) has shown that a way to solve some problems for quarks is to consider them as pF. This possibility is still seriously considered among others (e.g. Gell-Mann, 1972; for related work see Bracken & Green, 1973). This shows that the quantised version of Nambu's new Hamiltonian Mechanics may be suitable for the quark model.

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§ Green (1953), Greenberg & Messiah (1965).